–Consider a simple ecosystem consisting of rabbits that have an infinite food supply and foxes that prey upon the rabbits for their food. A classical mathematical model due to Volterra describes this system by a pair of nonlinear first-order equations:

$$\begin{aligned} dr/dt &= 2r - \alpha r f, \quad & r(0) &= r_0, \\ df/dt &= -f + \alpha r f, \quad & f(0) &= f_0, \end{aligned}$$

where t is time, $r = r(t)$ is the number of rabbits, $f = f(t)$ is the number of foxes, and α is a positive constant. When $\alpha = 0$, the two populations do not interact, and the rabbits do what rabbits do best and the foxes die off from starvation. When $\alpha > 0$, the foxes encounter the rabbits with a probability which is proportional to the product of their numbers. Such an encounter results in a decrease in the number of rabbits and (for less obvious reasons) an increase in the number of foxes.

Investigate the behavior of this system for $\alpha = 0.01$ and various values of r_0 and f_0 ranging from 2 or 3 to several thousand. Draw graphs of interesting solutions. Include a phase plane plot. Since we are being rather vague about the units of measurement, there is no reason to restrict r and f to integer values.

(a) Compute the solution with $r_0 = 300$ and $f_0 = 150$. You should observe from the output that the system is periodic with a period close to five time units. In other words $r(0) \approx r(5)$ and $f(0) \approx f(5)$.

(b) Compute the solution with $r_0 = 15$ and $f_0 = 22$. You should find that the number of rabbits eventually drops below 1. This could be interpreted by saying that the rabbits become extinct. Find initial conditions which cause the foxes to become extinct. Find initial conditions with $r_0 = f_0$ which cause both species to become extinct.

(c) Is it possible for either component of the true solution to become negative? Is it possible for the numerical solution to become negative? What happens if it does? (In practice, the answers to the last two questions may depend on the value used for your error tolerance.)

(d) Many modifications of this simple model have been proposed to more accurately reflect what happens in nature. For example, the number of rabbits may be prevented from growing indefinitely by changing the first equation to

$$\frac{dr}{dt} = 2\left(1 - \frac{r}{R}\right) f - \alpha r f.$$

Now, even when $\alpha = 0$, the number of rabbits can never exceed R. Pick some reasonable value of R, and consider some of the above questions again. In particular, what happens to the periodicity of the solutions?

This model has been studied extensively by both mathematicians and biologists. Comparisons with actual observations of lynx and hare populations in the Hudson Bay area have even been made. Many interesting properties are described in the readable book by Haberman (1977).